ISSN 2229-5518

# Three-Point Boundary Value Problems On Fuzzy Differential Equations- Existence and Uniqueness.

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**Abstract**—Methods of fuzzy differential equations are extending to include Variation of Parameters formula, Existence and Uniqueness criteria for Three point Boundary Value Problems. Incidentally, we prove existence and uniqueness criteria for Initial Value Problems On fuzzy differential equations.

**Index Terms**— Differentiable mapping, Lipchitz condition, Fuzzy Boundary Value Problem, Green's matrix.

## 1 INTRODUCTION

In recent years the theory of Fuzzy Differential Equations got more attention because this theory represents the natural way of modeling the dynamical systems under uncertainty. Green's function plays a vital role in solving Boundary Value Problems of fuzzy differential equations.

in this paper we consider a system of first order in-homogeneous differential equation

$$y'(t) = A(t)y(t) + f(t), \quad y(t_0) = y_0, \qquad (1.1)$$

where $y$ is n- dimensional fuzzy vector, A is an $n \times n$ vector valued function. The concept of fuzzy derivative was first introduced by Chang and Zadhe [2]. Initially the derivative of fuzzy valued mapping was developed by Puri and Ralescu [9], that generalized and extended the concept of Hukuhara differentiability (H-derivative) for set valued mappings. Using H-derivative, Kaleva [3] develop a theory for fuzzy differential equations. In order to provide a fundamental background, we refer to work of J.J. Nieto [6], and V.Lakshmikantham and R.N. Mohapatra [5]. In Kaleva [3] existence and uniqueness is developed under the Lipchitz condition. In [3] Lipchitz condition is replaced by more general condition and studied existence and uniqueness of initial value problems and established the global existence of solutions assuming local existence. Nonlinear Two Point Boundary Value Problems on Fuzzy Differential Equations, Lakshmikantham and K.N. Murty [4], in the year 2008 Minghochen and Wu [8] obtain Existence and Uniqueness on Fuzzy Differential Equations gave an amendment to results of D.O′ Regan Lakshmikantham [7]. They proved boundary value problems on fuzzy differential equations can be obtained by the means of the theory of obstruct function in Banach Spaces. In fact they prove the two point boundary value problems on fuzzy differential equations to fuzzy integral equations and further prove same results about existence and uniqueness. However the theory is not extended to system of differential equations.

This paper presents several intricacies involved in understanding fuzziness in differential equations. In this paper we introduce the concept of H-differentiability of Puri and Relescu [9] and study the properties of differentiable mappings. In section 2, we recall some fundamental concepts of fuzzy numbers and derivatives In section3, we present the criteria for existence of the general solution of the homogeneous system $y'(t) = A(t)y(t)$ and then present variation of parameters in section 4, In section 5, we discuss the existence and uniqueness criteria for three-point boundary value problem associated with the system of first order linear matrix differential equations.

## 2 PRELIMINARIES

The family of all non-empty compact convex subset of $\mathbb{R}^n$ denoted by $P_k(\mathbb{R}^n)$.

For $\alpha, \beta \in \mathbb{R}$ and $A, B \in P_k(\mathbb{R}^n)$, we define

$$\alpha(A + B) = \alpha A + \alpha B, \ \alpha(\beta A) = (\alpha\beta)A \text{ and } 1.A = A,$$

If $\alpha, \beta \geq 0$, then $(\alpha + \beta)A = \alpha A + \beta A$.

Let $T = [a, b]$ be a compact subinterval of $\mathbb{R}$.

Definition 21. Let $E^n = \{u: \mathbb{R}^n \to [0,1]\}, u \in E^n$ is called fuzzy number if it satisfies the following four axioms.

(i) $u$ is normal, that is there exists an $x_0 \in \mathbb{R}^n$ such that $u(x_0) = 1$.

(ii) $u$ is fuzzy convex, that is for any $x, y \in \mathbb{R}^n$ and $0 < \lambda < 1$, $u(\lambda x + (1-\lambda)y) \in \mathbb{R}^n$.

(iii) $u$ is upper semi continuous.

(iv) $[u]^\alpha = \{x \in \mathbb{R}^n : u(x) \geq \alpha\}$. .

For $0 \leq \alpha \leq 1$, the $\alpha$- level sets $[u]^\alpha \in P_k(\mathbb{R}^n)$.

We note that $[u]^0 = \{x \in \mathbb{R}^n : u(x) \geq 0\}$ is compact.

Definition 2.2. A fuzzy number in parametric form is represented by $(u_\alpha^-, u_\alpha^+)$, where

$$u_\alpha^- = \min[u]^\alpha, \quad u_\alpha^+ = \max[u]^\alpha \ , 0 \leq \alpha \leq 1$$

and has the following properties.

(i) $u_\alpha^-$ is a bounded left-continuous monotone increasing function of $\alpha$ over $[0,1]$

(ii) $u_\alpha^+$ is bounded left- continuous monotone decreasing function of $\alpha$ over $[0,1]$

(iii) $u_\alpha^- \leq u_\alpha^+$ , $0 \leq \alpha \leq 1$.

If $f: \mathbb{R}^n \to \mathbb{R}^n$ is a function, then according to Zadeh's extension principle, we can extend $f: E^n \times E^n \to E^n$ by defining

$$f(u,v)(z) = \sup_{z=f(x,y)} \min(u(x), v(y)) \quad \text{and} \tag{2.1}$$

it is well known that

$$[f(u,v)]^\alpha = f([u]^\alpha, [v]^\alpha) \tag{2.2}$$

For all $u, v \in E^n$, $\lambda \in \mathbb{R}$ and $0 \leq \alpha \leq 1$, the sum $u + v$ and the product $\lambda u$ are defined by

$$[u+v]^\alpha = [u]^\alpha + [v]^\alpha$$

$$[\lambda u]^\alpha = \lambda [u]^\alpha$$

where $[u]^\alpha + [v]^\alpha$ means the usual addition of two intervals (subsets) of $\mathbb{R}^n$ and $\lambda[u]^\alpha$ means the usual product between a scalar and subset of $\mathbb{R}^n$.

Definition 2.3. We define $D: E^n \times E^n \to \mathbb{R}_+ \cup \{0\}$ by

$$D(u,v) = \sup_{0 \leq \alpha \leq 1} d_H([u]^\alpha, [v]^\alpha),$$

where $d_H$ is the Hausdorf metric defined in $P_k(\mathbb{R}^n)$

$$\text{i.e. } D(u,v) = \sup_{0 \leq \alpha \leq 1} \max\{|u_\alpha^-, v_\alpha^-|, |u_\alpha^+, v_\alpha^+|\}.$$

It can easily verify that $(E^n, D)$ is a complete metric space and that $D$ has the following properties:

For any $u, v, w \in P_k(\mathbb{R}^n)$ and $\lambda \in \mathbb{R}$

(i) $D(u+w, v+w) = D(u,v)$

(ii) $D(\lambda u, \lambda v) = |\lambda| D(u,v)$

(iii) $D(u,v) \leq D(u,w) + D(w,v)$.

Definition 2.4. Let $x, y \in E^n$. If there exists $z \in E^n$ such that $x = y + z$, then $z$ is called the H-difference of $x$ with respect of $y$ and is denoted by $x \ominus y$.

## 3 Differentiation and Integration of Fuzzy Number Value Functions

Definition 3.1. Let $F: T \to E^n$. For $t_0 \in T$, we say that $F$ is differentiable at $t_0$ (H- differentiable), if there exists an element $F'(t_0) \in E^n$ such that for all $h > 0$, the H- difference $F(t_0 + h) \ominus F(t_0)$, $F(t_0) \ominus F(t_0 - h)$ exists and the limits (in the metric D)

$$\lim_{h \to 0^+} \frac{F(t_0+h) \ominus F(t_0)}{h}, \quad \lim_{h \to 0^+} \frac{F(t_0) \ominus F(t_0-h)}{h}$$

are exists and each equals to $F'(t_0)$.

At the end points of $T$, we only take one-sided derivative.

Proposition 3.1. Let $f: T \to E^n$ be continuous on $T$, the it is integrable on $T$.

Proposition 3.2. Let $f: T \to E^n$ be integrable on $[a,b]$, $a < c < b$, then $f$ is integrable on $[a,c]$ and $[c,b]$.

$$i.e. \int_a^b f(t)dt = \int_a^c f(t)dt + \int_c^b f(t)dt.$$

Proposition 3.3. Let $f, g: T \to E^n$ be integrable on $T$ and $\alpha, \beta \in \mathbb{R}$, then

$$\int_a^b (\alpha f(t) + \beta g(t))dt = \alpha \int_a^b f(t)dt + \beta \int_a^b g(t)dt.$$

Proposition 3.4. If $g(t) = \int_{t_0}^t f(t)dt$ is differentiable and $g'(t) = f(t)$. Then

ISSN 2229-5518

$$f(t) - f(t_0) = \int_{t_0}^{t} f'(s)ds.$$

**Theorem 3.1.** Let $F: T \to E^1$ be differentiable. Denote $F_\alpha(t) = [f_\alpha(t), g_\alpha(t)]$ for $\alpha \in [0,1]$. Then $f_\alpha(t)$ and $g_\alpha(t)$ are differentiable and

$$[F'(t)]^\alpha = [f'_\alpha, g'_\alpha].$$

**Proof.** Clearly for any $\alpha \in [0,1]$,

$$[F(t+h) - F(t)]^\alpha = [f_\alpha(t+h) - f_\alpha(t), g_\alpha(t+h) - g_\alpha(t)]$$

and

$$[F(t) - F(t-h)]^\alpha = [f_\alpha(t) - f_\alpha(t-h), g_\alpha(t) - g_\alpha(t-h)]$$

$$\lim_{h\to 0^+}\left[\frac{F(t+h) - F(t)}{h}\right]^\alpha = \lim_{h\to 0^+}\left[\frac{f_\alpha(t+h) - f_\alpha(t)}{h}\right]^\alpha, \lim_{h\to 0^+}\left[\frac{g_\alpha(t+h) - g_\alpha(t)}{h}\right]^\alpha$$

and

$$\lim_{h\to 0^+}\left[\frac{F(t) - F(t-h)}{h}\right]^\alpha = \lim_{h\to 0^+}\left[\frac{f_\alpha(t) - f_\alpha(t-h)}{h}\right]^\alpha, \lim_{h\to 0^+}\left[\frac{g_\alpha(t) - g_\alpha(t-h)}{h}\right]^\alpha$$

Hence

$$[F'(t)]^\alpha = [f'_\alpha, g'_\alpha].$$

**The following theorems are immediate.**

**Theorem 3.2.** If $F: T \to E^n$ be differentiable at t, then it is continuous at t.

**Theorem 3.3.** If $F, G: T \to E^n$ are differentiable and $\lambda \in \mathbb{R}$, then

$$(F+G)'(t) = F'(t) + G'(t)$$

and

$$(\lambda F)'(t) = \lambda F'(t).$$

**Theorem 3.4.** Let $F: T \to E^n$ be continuous. Then for all $t \in T$,

$G(t) = \int_a^t F$ is differentiable and $G'(t) = F(t)$.

Fuzzy differential equations

Definition 3.3. Let $\alpha > 0$ and $f: T \times E^n \to E^n$ be continuous, we say that f satisfies a Lipchitz condition with the Lipchitz constant $K > 0$, if for any $(t, x), (t, y) \in T \times E^n$,

$$D(f(t,x), f(t,y)) \le K\, D(x,y) \qquad (3.1)$$

**Lemma 3.1.** $\emptyset : T \to E^n$ is a solution of the initial value problem

$$x' = f(t,x), x(t_0) = x_0 \qquad (3.2)$$

if and only if it is a solution of the integral equation

$$x(t) = x_0 + \int_t^{t_0} f(s, x(s))\, ds$$

for all $t \in T$ and $t > t_0$.

**Theorem 3.5.** Let $f: T \times E^n \to E^n$ be continuous and assume that of satisfies a Lipchitz condition (3.1) with the Lipchitz constant $K > 0$. Then the initial value problem (3.2) has one and only one solution on $T$.

**Proof.** For any $\emptyset, \psi \in C(J, E^n)$, define

$$H(\emptyset, \psi) = \sup_{t \in J} D(\emptyset(t), \psi(t))$$

Since $(E^n, D)$ is a complete metric space it follows that $C(J, E^n)$ is a complete metric space.

Now let $(t, y_1) \in T \times E^n$ and $\eta > 0$ be such that $\eta k < 1$.

Then consider

$$T\emptyset(t) = y_1 + \int_{t_1}^{t} f(s, \emptyset(s))\, ds$$

Using the fact that f satisfy a Lipchitz condition ,

We have

$$H(T\emptyset, T\psi) = \sup_{t \in J} D\left(\int_{t_1}^{t} f(s, \emptyset(s))ds, \int_{t_1}^{t} f(s, \psi(s))\, ds\right)$$

$$\le \int_{t1}^{t_1+\eta} D\big(f(s, \emptyset(s)),\ f(s, \psi(s))\big)\, ds$$

$$\le \int_{t1}^{t_1+\eta} k\, D\big(\emptyset(s),\ \psi(s)\big)\, ds$$

$$\le \eta\, K\, H(\emptyset, \psi).$$

ISSN 2229-5518

for all $\emptyset, \psi \in C[J, E^n]$. Hence by the Generalized contraction mapping theorem T has a unique fixed point wherever $\eta k < 1$, which is in fact the desired solution of the initial value problem .

## 4 Variation of Parameters Formula

We now our attention to the linear system

$$x' = A(t)x + f(t)$$

where $A, f: T \rightarrow E'$ are continuous. If $A_\alpha(t) = [a_1^\alpha(t),\ a_2^\alpha(t)]$

and $[x]^\alpha = [x_1^\alpha, x_2^\alpha]$, then

$$[A(t)x]^\alpha = \left[\min_{t\in[a,b]}(a_1^\alpha(t)x_1, a_2^\alpha(t)x_2, a_2^\alpha(t)x_2^\alpha, a_2^\alpha(t)x_2^\alpha), \max_{t\in[a,b]}(a_1^\alpha(t)x_1, a_2^\alpha(t)x_2, a_2^\alpha(t)x_2^\alpha, a_2^\alpha(t)x_2^\alpha)\right]$$

Since the components of $A_\alpha(t)$ are continuous on a closed interval, they are bounded on T and hence there exists a constant $K > 0$ such that $D(A_\alpha(t)x, 0) \leq KD(x(t), 0)$.

By the uniqueness of initial value problems, the initial value problem

$$y' = A(t)y + f(t), \quad y(t_0) = y_0$$

has unique solution on T.

We use the following notations:

Consider $y' = A(t)y + f(t)$

Where $y = (y_1, y_2, \dots y_n)$ and each $y_i \in E^n$ and hence $y \in E^{nn}$,

Define for each $\alpha \in [0,1]$

$$u_1^\alpha = [a_{11}u_1 + a_{12}, u_2 + \dots + a_{1n}u_n]^\alpha$$

$$= \sup_{t\in(0,1)}[a_{11}[u_1]^\alpha + a_{12}[u_2]^\alpha + \dots a_{1n}[u_n]^\alpha]$$

and $D(x,y) = [d(x_1y_1), d(x_2y_2), \dots, d(x_ny_n)]$ as generalize metric.

Note that the system given order non-homogeneous fuzzy differential equation

$$y' = A(t)y + f(x) \tag{4.1}$$

is equivalent to the following system of first order non-homogeneous fuzzy differential equations

$$(y_\alpha^-)' = Ay_\alpha^-(t) + f_\alpha^-(t) \tag{4.2}$$

$$(y_\alpha^+)' = Ay_\alpha^+(t) + f_\alpha^+(t) \tag{4.3}$$

Where $y_\alpha^\pm(t) = \left(y_{1\alpha}^\pm(t), \dots, y_{n\alpha}^\pm(t)\right)^T$ and

$$f_\alpha^\pm(t) = \left(f_{1\alpha}^\pm(t), \dots, f_{n\alpha}^\pm(t)\right).$$

The general solution of the homogeneous systems

$(y_\alpha^-)'(t) = Ay_\alpha^-(t)$ satisfying $y_\alpha^-(t_0) = y_{0,\alpha}^-$

when A is a constant matrix can formally be expressed as

$$y_\alpha^-(t) = \exp(-A(t - t_0))\, y_{0,\alpha}^-$$

and the general solution of

$(y_\alpha^+)'(t) = Ay_\alpha^+(t)$, Satisfying $(y_\alpha^+)(t_0) = y_\alpha^+(t)$ is

$$y_\alpha^+(t) = \exp\left(-A(t - t_0)\right) y_{0,\alpha}^+ .$$

If $Y(t)$ is a fundamental matrix of $y' = A(t)y$, then any solution of Non-homogeneous equation (4.2) and (4.3) are given by

$$(y_\alpha^-)(t) = Y_\alpha^-(t)\, y_{0,\alpha}^-(t), + Y_\alpha^-(t) \int_0^t Y_\alpha^-(s)\, f_\alpha^-(s)\, ds \tag{4.4}$$

and

$$(y_\alpha^+)(t) = Y_\alpha^+(t)\, y_{0,\alpha}^+ + Y_\alpha^+(t) \int_0^t Y_\alpha^+(s)\, f_\alpha^+(s)\, ds \tag{4.5}$$

## 5 Existence and Uniqueness

In this section, we consider the general boundary value problem

$$y'(t) = A(t)y + f(t), \quad a \leq t \leq c$$

$$My(a) + Ny(b) + Ry(c) = 0 \quad (a < b < c) \tag{5.1}$$

Where M, N and R are constant square matrices of order 'n 'and all scalars are assumed to be real.

The boundary value problem ( 5.1) is equivalent to the following system of fuzzy boundary value problems for $\alpha \in [0,1]$ and for every $t \in T$.

$$(y_\alpha^-)'(t) = Ay_\alpha^-(t) + f_\alpha^-(t)$$

$$My_\alpha^-(a) + Ny_\alpha^-(b) + Ry_\alpha^-(c) = 0 \qquad (5.2)$$

and

$$(y_\alpha^+)'(t) = Ay_\alpha^+(t) + f_\alpha^+(t)$$

$$My_\alpha^+(a) + Ny_\alpha^+(b) + + Ry_\alpha^+(c) = 0 \qquad (5.3)$$

Definition 5.1. If $(Y_\alpha^\pm)(t)$ is a fundamental matrix solution of $(y_\alpha^\pm)'(t) = A(t)(y_\alpha^\pm)(t)$,

then the matrix D defined by

$$D_\alpha^\pm = MY_\alpha^\pm(a) + NY_\alpha^\pm(b) + RY_\alpha^\pm(c)$$

is called characteristic matrices for the boundary value problem

Definition 5.2. A boundary value problem is said to be incompatible, if its index of compatibility is zero.

i.e. $\quad D_\alpha^- = MY_\alpha^-(a) + NY_\alpha^-(b) + RY_\alpha^-(c)$

and $\quad D_\alpha^+ = MY_\alpha^+(a) + NY_\alpha^+(b) + RY_\alpha^+(c)$

are Non-singular.

Theorem 5.1. The boundary value problem

$$(y_\alpha^-)'(t) = Ay_\alpha^-(t) + f_\alpha^-(t)$$

$$My_\alpha^-(a) + Ny_\alpha^-(b) + Ry_\alpha^-(c) = 0$$

and

$$(y_\alpha^+)'(t) = Ay_\alpha^+(t) + f_\alpha^+(t)$$

$$My_\alpha^+(a) + Ny_\alpha^+(b) + + Ry_\alpha^+(c) = 0$$

has unique solution, which is

$$y_\alpha^-(t) = \int_a^c G_\alpha^-(t,s) f_\alpha^-(s)\, ds$$

and

$$y_\alpha^+(t) = \int_a^c G_\alpha^+(t,s) f_\alpha^+(s)\, ds$$

where $G_\alpha^\pm(t,s)$ are the green's matrices for the homogeneous boundary value problem

$$(y_\alpha^\pm)' = A(t)\, y_\alpha^\pm(t)$$

Proof:

Substituting the general form of $y_\alpha^-$ and $y_\alpha^+$ as given (4.4) and (4.5) in the boundary condition matrices of ( 5.2 ) and ( 5.3 ) respectively, we get

$$[\, MY_\alpha^-(a) + NY_\alpha^-(b) + RY_\alpha^-(c)\,]y_{0,\alpha}^- + NY_\alpha^-(b)\int_a^b (Y_\alpha^-)^{-1}(s) f_\alpha^-(s)\, ds + RY_\alpha^-(c)\int_a^c (Y_\alpha^-)^{-1}(s) f_\alpha^-(s)\, ds = 0$$

and

$$[\, MY_\alpha^+(a) + NY_\alpha^+(b) + RY_\alpha^+(c)\,]y_{0,\alpha}^+ + NY_\alpha^+(b)\int_a^b (Y_\alpha^+)^{-1}(s) f_\alpha^+(s)\, ds + RY_\alpha^+(c)\int_a^c (y_\alpha^+)^{-1}(s) f_\alpha^+(s)\, ds = 0$$

$$y_{0,\alpha}^-(\alpha) = -(D_\alpha^-)^{-1}\left[NY_\alpha^-(b)\int_a^b (Y_\alpha^-)^{-1}(s) f_\alpha^- ds + RY_\alpha^-(c)\int_a^c (Y_\alpha^-)^{-1}(s) f_\alpha^-(s) ds\right]$$

and

$$y_{0,\alpha}^+(\alpha) = -(D_\alpha^+)^{-1}\left[NY_\alpha^+(b)\int_a^b (Y_\alpha^+)^{-1}(s) f_\alpha^+ ds + RY_\alpha^+(c)\int_a^c (Y_\alpha^+)^{-1}(s) f_\alpha^+(s) ds\right]$$

Substituting $y_{0,\alpha}^\pm$ in the general form we get

$$y_\alpha^-(t) = Y_\alpha^-(t)\int_a^t (Y_\alpha^-)^{-1}(s) f_\alpha^-(s) ds - Y_\alpha^-(t)(D_\alpha^-)^{-1}\left[NY_\alpha^-(b)\int_a^b (Y_\alpha^-)^{-1}(s) f_\alpha^- ds + RY_\alpha^-(c)\int_a^c (Y_\alpha^-)^{-1}(s) f_\alpha^-(s) ds\right]$$

and

$$y_\alpha^+(t) = Y_\alpha^+(t)\int_a^t (Y_\alpha^+)^{-1}(s) f_\alpha^+(s) ds - Y_\alpha^+(t)(D_\alpha^+)^{-1}\left[NY_\alpha^+(b)\int_a^b (Y_\alpha^+)^{-1}(s) f_\alpha^+ ds + RY_\alpha^+(c)\int_a^c (Y_\alpha^+)^{-1}(s) f_\alpha^+(s) ds\right]$$

It can be written as

$$y_\alpha^-(t) = \int_a^c G_\alpha^-(t,s) f_\alpha^-(s)\, ds$$

and

$$y_\alpha^+(t) = \int_a^c G_\alpha^+(t,s) f_\alpha^+(s)\, ds$$

where $G_\alpha^\pm(t,s)$ are the green's matrices for the homogeneous boundary value problem

$$(y_\alpha^\pm)' = A(t)\, y_\alpha^\pm(t)$$

Satisfying the boundary condition matrix

$$My_\alpha^\pm(a) + Ny_\alpha^\pm(b) + Ry_\alpha^\pm(c) = 0$$

and is given by

$$\underset{t\in[a,b]}{G_\alpha^-(t,s)} = \begin{cases} Y_\alpha^-(t)(D_\alpha^-)^{-1} MY_\alpha^-(a)(Y_\alpha^-)^{-1}(s), & a < s < t \le b < c \\ -Y_\alpha^-(t)(D_\alpha^-)^{-1}[NY_\alpha^-(b) + RY_\alpha^-(c)](Y_\alpha^-)^{-1}(s), & a \le t < s < b < c \\ -Y_\alpha^-(t)(D_\alpha^-)^{-1} RY_\alpha^-(c)(Y_\alpha^-)^{-1}(s), & a < t < b < s < c. \end{cases}$$

$$\underset{t\in[b,c]}{G_\alpha^-(t,s)} = \begin{cases} -Y_\alpha^-(t)(D_\alpha^-)^{-1} RY_\alpha^-(c)(Y_\alpha^-)^{-1}(s), & a < b < t < s < c \\ Y_\alpha^-(t)(D_\alpha^-)^{-1}[MY_\alpha^-(a) + NY_\alpha^-(b)](Y_\alpha^-)^{-1}(s), & a < b < s < t \le c \\ Y_\alpha^-(t)(D_\alpha^-)^{-1} MY_\alpha^-(a)(Y_\alpha^-)^{-1}(s), & a < s < b \le t < c. \end{cases}$$

and

$$\underset{t\in[a,b]}{G_\alpha^+(t,s)} = \begin{cases} Y_\alpha^+(t)(D_\alpha^+)^{-1} MY_\alpha^+(a)(Y_\alpha^+)^{-1}(s), & a < s < t \le b < c \\ -Y_\alpha^+(t)(D_\alpha^+)^{-1}[NY_\alpha^+(b) + RY_\alpha^+(c)](Y_\alpha^+)^{-1}(s), & a \le t < s < b < c \\ -Y_\alpha^+(t)(D_\alpha^+)^{-1} RY_\alpha^+(c)(Y_\alpha^+)^{-1}(s), & a < t < b < s < c. \end{cases}$$

$$\underset{t\in[b,c]}{G_\alpha^+(t,s)} = \begin{cases} -Y_\alpha^+(t)(D_\alpha^+)^{-1} RY_\alpha^+(c)(Y_\alpha^+)^{-1}(s), & a < b < t < s < c \\ Y_\alpha^+(t)(D_\alpha^+)^{-1}[MY_\alpha^+(a) + NY_\alpha^+(b)](Y_\alpha^+)^{-1}(s), & a < b < s < t \le c \\ Y_\alpha^+(t)(D_\alpha^+)^{-1} MY_\alpha^+(a)(Y_\alpha^+)^{-1}(s), & a < s < b \le t < c. \end{cases}$$

Theorem 5.2. Green matrix $G_\alpha^\pm(t,s)$ has been following properties:

(i) The components of $G_\alpha^\pm(t,s)$ regarded as a foundation 't' for a fixed 's' is continuous on $[a,s)$ and $(s,b]$. At the point $t = s$, $G_\alpha^\pm$ has an upward jump discontinuity of unit magnitude .

i.e. $[G_\alpha^\pm(s^+,s) - G_\alpha^\pm(s^-,s)] = I_n$

(ii) $G_\alpha^\pm$ is a formal solution of the homogeneous boundary value problems , it fails to be a true solution become of the discontinuity at $t = s$.

(iii) $G_\alpha^\pm$ is a unique $n \times n$ matrix with properties (i) & (ii)

Proof : we prove the properties in the following cases. The other case follows similar

Suppose $s \in [a,b]$

$$G_\alpha^\pm(s^+,s) - G_{,\alpha}^\pm(s^-,s) = Y_\alpha^\pm(s)(D_\alpha^\pm)^{-1} MY_\alpha^\pm(a)\,(Y_\alpha^\pm)^{-1}(s)$$
$$+Y_\alpha^\pm(s)\,(D_\alpha^\pm)^{-1}[NY_\alpha^\pm(b) + RY_\alpha^\pm(c)](Y_\alpha^\pm)^{-1}(s)$$
$$= Y_\alpha^\pm(s)(D_\alpha^\pm)^{-1}[MY_\alpha^\pm(a) + NY_\alpha^\pm(b) + RY_\alpha^\pm(c)](Y_\alpha^\pm)^{-1}(s)$$
$$= Y_\alpha^\pm(s)(D_\alpha^\pm)^{-1} D_\alpha^\pm\,(Y_\alpha^\pm)^{-1}(s)$$
$$= I_n$$

Hence property ( i ) proved .

To prove the property ( i i ), consider

$$MG_\alpha^\pm(a,s) + NG_\alpha^\pm(b,s) + RG_\alpha^\pm(c,s)$$
$$= -MY_\alpha^\pm(a)(D_\alpha^\pm)^{-1}[NY_\alpha^\pm(b) + RY_\alpha^\pm(c)](Y_\alpha^\pm)^{-1}(s)$$
$$+N[Y_\alpha^\pm(b) - Y_\alpha^\pm(b)(D_\alpha^\pm)^{-1} NY_\alpha^\pm(b) - Y_\alpha^\pm(b)(D_\alpha^\pm)^{-1} RY_\alpha^\pm(c)](Y_\alpha^\pm)^{-1}(s)$$
$$+R[Y_\alpha^\pm(c) - Y_\alpha^\pm(c)(D_\alpha^\pm)^{-1} NY_\alpha^\pm(b) - Y_\alpha^\pm(c)(D_\alpha^\pm)^{-1} RY_\alpha^\pm(c)](Y_\alpha^\pm)^{-1}(s)$$
$$= -[MY_\alpha^\pm(a) + NY_\alpha^\pm(b) + RY_\alpha^\pm(c)]\,(D_\alpha^\pm)^{-1} NY_\alpha^\pm(b)Y_\alpha^\pm(s)$$
$$-[MY_\alpha^\pm(a) + NY_\alpha^\pm(b) + RY_\alpha^\pm(c)]\,(D_\alpha^\pm)^{-1} RY_\alpha^\pm(c)Y_\alpha^\pm(s)$$
$$+ NY_\alpha^\pm(b)(y_\alpha^\pm)^{-1}(s) + RY_\alpha^\pm(c)(Y_\alpha^\pm)^{-1}(s)$$
$$= 0.$$

Hence property ( ii) is proved.

finally , any solution of the non-homogeneous system

$$y' = A(t)y + f(t), \qquad a \le t \le c$$

$$My(a) + Ny(b) + Ry(c) = 0$$

is given by

$$y(t) = \int_a^c G(t,s)f(s)ds$$

where

$$y_\alpha(t) = \int_a^c G_\alpha(t,s)f_\alpha(s)ds$$

and

$$y(t) = \bigcup_{\alpha\in(0,1)} [y_{1,\alpha}^-(t), y_{1,\alpha}^+(t)] \ldots \bigcup_{\alpha\in(0,1)} [y_{n,\alpha}^-(t), y_{n,\alpha}^+(t)]$$

where

$$y_{i,\alpha}(t) = \int_a^c G_{i,\alpha}(t,s)f_\alpha(s)ds\,.$$

## n-Point Boundary Value problem:

In this section we write the Green's matrix for n- point boundary value problem. Consider n-Point Boundary Value problem

$$y' = A\,(t)y + f(t), \quad x_1 \le t \le x_n$$

$$M_1\, y(x_1) + \;\ldots + M_n\, y(x_n) = \;0, \qquad (x_1 < x_2 < \cdots < x_n)$$

Where $M_1, M_2 \ldots, M_n$ are constant square matrices of order n. The above system is equivalent to following fuzzy boundary value problem for $\alpha \in [0,1]$ and for every $t \in T$.

$$(y_\alpha^-)'(t) = A(t)y_\alpha^-(t) + f_\alpha^-(t)$$

$$M_1\, y_\alpha^-(x_1) + M_2\, y_\alpha^-(x_2) + \;\ldots + M_n y_\alpha^-(x_n) = \;0 \quad (5.4)$$

$$(y_\alpha^+)'(t) = Ay_\alpha^+(t) + f_\alpha^+(t)$$

$$M_1\, y_\alpha^+(x_1) + M_2\, y_\alpha^+(x_2) + \;\ldots + M_n y_\alpha^+(x_n) = \;0 \quad (5.5)$$

The solution of the above system is

$$y_\alpha^-(t) = \int_{x_1}^{x_n} G_\alpha^-(t,s)f_\alpha^-(s)\, ds$$

and

$$y_\alpha^+(t) = \int_{x_1}^{x_n} G_\alpha^+(t,s)f_\alpha^+(s)\, ds.$$

where $G_\alpha^\pm(t,s)$ are the green's matrices for the homogeneous boundary value problem

$$(y_\alpha^\pm)' = A(t)\, y_\alpha^\pm(t)$$

Satisfying the boundary condition matrix

$$M_1\, y_\alpha^\pm(x_1) + M_2\, y_\alpha^\pm(x_2) + \;\ldots + M_n y_\alpha^\pm(x_n) = \;0$$

and is given by

$$\underset{t\in[x_1,x_2]}{G_\alpha^\pm(t,s)} = \begin{cases} Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} M_1 Y_\alpha^\pm(x_1)(Y_\alpha^\pm)^{-1}(s), & x_1 < s < t \le x_2 < x_3 < \cdots < x_n \\ -Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} [M_2 Y_\alpha^\pm(x_2) + \cdots + M_n Y_\alpha^\pm(x_n)](Y_\alpha^\pm)^{-1}(s), & x_1 \le t < s < x_2 < x_3 < \cdots < x_n \\ -Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} [M_3 Y_\alpha^\pm(x_3) + \cdots + M_n Y_\alpha^\pm(x_n)](Y_\alpha^\pm)^{-1}(s), & x_1 < t < x_2 < s < x_3 < \cdots < x_n\,. \\ \quad\cdots & \\ \quad\cdots & \\ \quad\cdots & \\ -Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} M_n Y_\alpha^\pm(x_n)(Y_\alpha^\pm)^{-1}(s), & x_1 < t < x_2 < x_3 < \cdots < x_{n-1} < s < x_n \end{cases}$$

$$\underset{t\in[x_2,x_3]}{G_\alpha^\pm(t,s)} = \begin{cases} Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} [M_1 Y_\alpha^\pm(x_1) + M_2 Y_\alpha^\pm(x_2)](Y_\alpha^\pm)^{-1}(s), & x_1 < x_2 < s < t \le x_3 < \cdots < x_n \\ -Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} [M_3 Y_\alpha^\pm(x_3) + \cdots + M_n Y_\alpha^\pm(x_n)](Y_\alpha^\pm)^{-1}(s), & x_1 < x_2 \le t < s < x_3 < \cdots < x_n \\ -Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} [M_4 Y_\alpha^\pm(x_2) + \cdots + M_n Y_\alpha^\pm(x_n)](Y_\alpha^\pm)^{-1}(s), & x_1 < x_2 < t < x_3 < s < x_4 < \cdots < x_n. \\ \quad\cdots & \\ \quad\cdots & \\ \quad\cdots & \\ -Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} M_n Y_\alpha^\pm(x_n)(Y_\alpha^\pm)^{-1}(s), & x_1 < x_2 < t < x_3 < \cdots < x_{n-1} < s < x_n \\ Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} M_1 Y_\alpha^\pm(x_1)(Y_\alpha^\pm)^{-1}(s), & x_1 < s < x_2 < t < x_3 < \cdots < x_{n-1} < x_n \end{cases}$$

$$\underset{t\in[x_{n-1},x_n]}{G_\alpha^\pm(t,s)} = \begin{cases} Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} [M_1 Y_\alpha^\pm(x_1) + \cdots + M_{n-1} Y_\alpha^\pm(x_{n-1})](Y_\alpha^\pm)^{-1}(s), & x_1 < x_2 < \cdots < x_{n-1} < s < t \le x_n \\ -Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} M_n Y_\alpha^\pm(x_n)(Y_\alpha^\pm)^{-1}(s), & x_1 < x_2 < \cdots < x_{n-1} \le t < s < x_n \\ Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} [M_1 Y_\alpha^\pm(x_1) + \cdots + M_{n-2} Y_\alpha^\pm(x_2)](Y_\alpha^\pm)^{-1}(s), & x_1 < x_2 < \cdots x_{n-2} < s < x_{n-1} < t < x_n \\ \quad\cdots & \\ \quad\cdots & \\ \quad\cdots & \\ Y_\alpha^\pm(t)(D_\alpha^\pm)^{-1} M_1 Y_\alpha^\pm(x_1)(Y_\alpha^\pm)^{-1}(s), & x_1 < s < x_2 < \cdots x_{n-1} < t < x_n \end{cases}$$

ISSN 2229-5518

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